On the languages accepted by finite reversible automata*

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Abstract

A reversible automaton is a finite (possibly incomplete) automaton in which each letter induces a partial one-to-one map from the set of states into itself. We give four non-trivial characterizations of the languages accepted by a reversible automaton equipped with a set of initial and final states and we show that one can effectively decide whether a given rational (or regular) language can be accepted by a reversible automaton. The first characterization gives a description of the subsets of the free group accepted by a reversible automaton that is somewhat reminiscent of Kleene's theorem. The second characterization is more combinatorial in nature. The decidability follows from the third — algebraic — characterization. The last and somewhat unexpected characterization is a topological description of our languages that solves an open problem about the finite-group topology of the free monoid.

1 Introduction.

The well-Known Kleene's theorem states that a language is rational if and only if it is accepted by a finite deterministic automaton. Since rational languages are closed under reversal, the rational languages are also exactly the languages accepted by *finite codeterministic automata* (an automaton $\mathcal{A}$ is codeterministic if the reverse automaton $\mathcal{A}^r$ is deterministic). The aim of this paper is to characterize the languages accepted by *reversible* (that is both deterministic and codeterministic) automata. Although this natural question requires only the very basic definitions of automata theory and

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could have been asked already in the fifties, the answer we propose in this paper is intimately related to the more advanced research on recognizable languages and finite semigroups.

We first need to make precise the meaning of the expression "accepted by an automaton". In the case of a deterministic automaton, one usually considers a set of final states and either a unique initial state, or a set of initial states : in both cases, the languages accepted by these automata are the rational languages. There are two solutions to keep the symmetry for the definition of a reversible automaton. We may consider a unique initial state and a unique final state: we call it the restricted mode of acceptance. Such acceptors have been considered in artificial intelligence in connection with the problem of inductively inferring general rules from examples [1]. They also have occurred in the study of the star-height problem [9] and are related to certain classes of biprefix codes [6]. The corresponding class of languages is not closed under union and the membership problem for this class is easy to solve. The second solution consists in considering a *set* of initial and final states. Now, the corresponding class of languages $\mathcal{C}$ is closed under (finite) union, but it is no longer trivial to decide whether or not a given rational language belongs to $\mathcal{C}$ (for instance, the minimal automaton of a language of $\mathcal{C}$ is not reversible in general). We propose in this paper four different characterizations of the class $\mathcal{C}$.

Our first characterization relates the class $\mathcal{C}$ to a class of rational subsets of the free group. Indeed, an automaton is reversible if and only if each letter induces a partial one-to-one map from the set of states into itself. Therefore, if a reversible automaton accepts a language L of A^* it also accepts in a natural way a subset K of the free group such that $L = K \cap A^*$. Now the subsets of the free group accepted by a reversible automaton form the smallest class of subsets (of the free group) containing the singletons and closed under the three operations "union", "product by an element of the free group", and "subgroup generated by". These subsets are also the finite unions of cosets of finitely generated subgroups of the free group.

Next, we observe that if a language L belongs to $\mathcal{C}$, then

(a) the idempotents of the syntactic monoid $M(L)$ commute.

This necessary condition plays an important role in semigroup theory [2, 3, 10, 11]. However, (a) is not sufficient, conversely, to ensure that L belongs to $\mathcal{C}$.

There are three different ways to strengthen condition (a) to obtain a characterization of $\mathcal{C}$. The first solution is to require that in L, "plus is equivalent to star", or more precisely, that

(b) if $xu^+y \in L$ for some words x, u, y of A^*, then $xy \in L$.

The second solution is an algebraic translation of (b) : if P denotes the image of L in $M(L)$, then

(c) for every $s, t \in M(L)$, and for every idempotent $e \in M(L)$, $set \in P$ implies $st \in P$.

This characterization is important because it shows that the membership problem for is decidable. Given a rational language L, one can effectively compute $R(L)$ and P and verify (a) and (c). Therefore, one may decide whether or not L belongs to $\mathcal{C}$.

Our last characterization relates $\mathcal{C}$ to the group topology of the free monoid [5, 12, 14]. This topology is defined by a distance in which, roughly speaking, two words are close if they are not distinguishable by a group of small cardinality. We show that a rational language L belongs to $\mathcal{C}$ if and only if $M(L)$ satisfies (a) and L is in this topology. In fact, we conjecture that a rational language L is closed if and only if the condition (c) holds. The results of this paper show that this conjecture is true if the idempotents of $M(L)$ commute. We refer the reader to [11] for the consequences of this conjecture in semigroup theory.

Some of the proofs are omitted and will appear elsewhere.

2 Reversible automata

Recall that a finite deterministic (incomplete) automaton is a triple $\mathcal{A} = (Q, A, .)$ where Q is a finite) set of states, A is the alphabet, and $(q, a) \to q \cdot a$ is a partial function from $Q \times A$ into Q. We denote by a $\mathcal{A}^r$ the reverse automaton of $\mathcal{A}$. In general $\mathcal{A}^r$ is a non-deterministic automaton. If it is deterministic, we say that $\mathcal{A}$ is *reversible* (or *injective* [6, 14]). Equivalently, $\mathcal{A}$ is reversible if every letter $a \in A$ induces an injective partial function from Q into itself. We say that a language L of A^* is accepted in the restricted mode by $\mathcal{A}$ if there exists an initial state i and a final state f such that $L = \{u \in A^* \mid i \cdot u = f\}$. We say that L is accepted by $\mathcal{A}$ if there exists a set I of initial states and a set F of final states such that

$$L = \{u \in A^* \mid \text{there exists } i \in I \text{ and } f \in F \text{ such that } i \cdot u = f\}$$

Example 2.1 The automaton represented in the following diagram is reversible and accepts the language $\{c, ac, bc\}$.

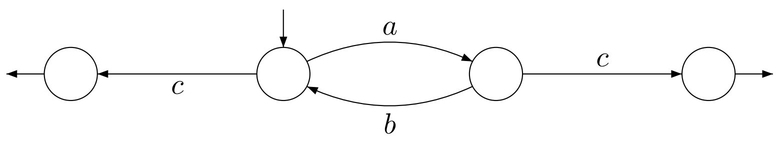

Figure 2.1: A reversible automaton accepting $\{c, ac, bc\}$.

It is not difficult to see that a trim, reversible automaton with a unique initial state and a unique final state is necessarily minimal. The next proposition, discovered independently by various authors, characterizes the languages accepted by reversible automaton in the restricted mode.

Proposition 2.1 *A language L is accepted by a reversible automaton in the restricted mode if and only if the minimal automaton of L is reversible and has a unique final state.*

It is much more difficult to characterize the class $\mathcal{C}$ of all the languages accepted by a reversible automaton. The following construction shows that $\mathcal{C}$ is closed under union. Given two reversible automata $\mathcal{A}_1 = (Q_1, A, \cdot_1, I_1, F_1)$ and $\mathcal{A}_2 = (Q_2, A, \cdot_2, I_2, F_2)$, we form the *disjoint union* $\mathcal{A}$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows: $\mathcal{A} = (Q, A, \cdot, I, F)$ where Q is the disjoint union of Q_1 and Q_2, I is the disjoint union of I_1 and I_2, F is the disjoint union of F_1 and F_2, and, for every letter $a \in A$,

$$q \cdot a = \begin{cases} q \cdot_1 a & \text{if } q \in Q_1 \\ q \cdot_2 a & \text{if } q \in Q_2 \end{cases}$$

Then $\mathcal{A}$ is a reversible automaton that accepts $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$.

Since the minimal automaton of a singleton $\{u\}$ is always reversible, it follows that $\mathcal{C}$ contains all the finite languages. Notice that the minimal automaton of a language of $\mathcal{C}$ is not necessarily reversible. For instance, the minimal automaton of $L = \{a, ac, bc\}$ is not reversible.

Let $\mathcal{A}$ be a reversible automaton. The action from A on Q can be extended to an action from the free group $F(A)$ on Q by setting, for every $q \in Q$, $q \cdot 1 = q$ and, for every $a \in A$ and for every $u \in A^*$,

$$q \cdot ua = (q \cdot u) \cdot a \text{if } q \cdot ua \text{ and } (q \cdot u) \cdot a \text{ are both defined (undefined otherwise)}$$
$$q \cdot u\bar{a} = q' \text{if } q \cdot u \text{ is defined and if } q' \cdot a = q \cdot u \text{ (undefined otherwise)}$$

Now, the subset of the free group accepted by $\mathcal{A}$ is the set

$$S(\mathcal{A}) = \{u \in F(A) \mid \text{there exists } q \in Q \text{ and } q' \in F \text{ such that } q \cdot u = q'\}$$

For instance, if $\mathcal{A}$ is the automaton of Example 2.1, then $S(\mathcal{A}) = \{c\} \cup \langle a\bar{b}\rangle\{ac, bc\}$ where $\langle X\rangle$ denotes the subgroup of $F(A)$ generated by a set X.

Since $L(\mathcal{A}) = S(\mathcal{A}) \cap A^*$, it suffices to describe the subsets of the free group accepted by a reversible automaton to obtain a first characterization of the class $\mathcal{C}$. This is the goal of the following theorem.

Theorem 2.2 *A subset S of the free group $F(A)$ is accepted by a reversible automaton if and only if S is a finite union of left cosets of finitely generated subgroups of the free group.*

(Proof omitted)

Here is another version of the same result, that is somewhat reminiscent of Kleene's theorem.

Theorem 2.3 *The subsets of the free group accepted by a reversible automaton form the smallest class of subsets $\mathcal{F}$ such that*

(1) *$\emptyset \in \mathcal{F}$ and for every $g \in F(A)$, $\{g\} \in \mathcal{F}$,*
(2) *if $S_1, S_2 \in \mathcal{F}$, then $S_1 \cup S_2 \in \mathcal{F}$,*
(3) *if $S \in \mathcal{F}$ and $g \in F(A)$ then $gS \in \mathcal{F}$,*
(4) *if $S \in \mathcal{F}$, then $\langle S\rangle \in \mathcal{F}$.*

3 An algebraic characterization of $\mathcal{C}$.

Let L be a rational language of A^*. We denote by $M(L)$ the syntactic monoid of L, by $\eta : A^* \to M(L)$ the syntactic morphism and we put $P = L\eta$. Recall that an element e of a monoid M is *idempotent* if $e = e^2$. The following proposition gives two important properties of the languages of $\mathcal{C}$.

Proposition 3.1 *If L is accepted by a reversible automaton, then,*

(a) *the idempotents of $M(L)$ commute,*
(b) *for every $x, u, y \in A^*$, $xu^+y \subset L$ implies $xy \in L$.*

Proof. Let $\mathcal{A} = (Q, A, \cdot, I, F)$ be a reversible automaton accepting L. Then L is recognized by the transition monoid M of $\mathcal{A}$. Let e be an idempotent of M. Then for every $q \in Q$, $(q\cdot e)\cdot e = q\cdot e$ whenever $q\cdot e$ is defined. Since $\mathcal{A}$ is reversible, e is an injective partial function and thus $q\cdot e = q$ or is undefined. In other words, every idempotent is a subidentity on Q. It follows immediately that the idempotents commute in M. Now, the syntactic monoid

$M(L)$ divides M (see [4, 7, 13] for instance), and since the class of monoids with commuting idempotents is closed under division, the idempotents also commute in $M(L)$.

Let $x, u, y \in A^*$ be words such that $xu^+y \subset L$. Since M is finite, there exists an integer $n > 0$ such that u^n is idempotent in M, that is, induces a subidentity on Q. Now since $xu^ny \in L$, there exists an initial state q and a final state q' such that $q \cdot xu^ny = q'$. Thus $(q \cdot x) \cdot u^n$ is defined, and hence is equal to $(q \cdot x)$. Therefore $q \cdot xy = q \cdot xu^ny = q'$ whence xy is accepted by $\mathcal{A}$ and thus $xy \in L$. ◻

Condition (b) of the previous proposition is equivalent to a more algebraic statement.

Proposition 3.2 *For every rational language L, the following conditions are equivalent:*

(b) *for every $x, u, y \in A^*$, $xu^+y \subset L$ implies $xy \in L$,*

(c) *for every $s, t \in M(L)$, and for every idempotent $e \in M(L)$, $set \in P$ implies $st \in P$.*

Proof. Assume that (b) is satisfied and let $s, e, t \in M(L)$ with e idempotent. Assume that $set \in P$. Then, since η is surjective, there exist some words $x, u, y \in A^*$ such that $x\eta = s$, $u\eta = e$ and $y\eta = t$. Now, for every $n > 0$, $(xu^ny)\eta = set \in P$. Thus $xu^+y \subset P\eta^{-1} = L$ and hence $xy \in L$ by (b). It follows that $st = (xy)\eta \in L\eta = P$.

Conversely, assume that (c) holds, and let x, u, y be words such that $xu^+y \subset L$. Then there exists $n > 0$ such that $u^n = e$ is an idempotent. Setting $x\eta = s$ and $y\eta = t$, we obtain $set \in L\eta = P$, and hence $st \in P$ by (c). Therefore $xy \in L$, since $(xy)\eta = st \in P$. ◻

We now turn to the converse of Proposition 3.1, for which we need a more detailed study of monoids with commuting idempotents. Recall that an element x of a monoid M is *regular* if there exists an element y such that $xyx = x$ and $yxy = y$. We start with an important combinatorial lemma, due to Ash [2].

Proposition 3.3 *Let M be a monoid with commuting idempotents, and let $\eta : A^* \to M$ be a monoid morphism. Then there exists an integer $N > 0$ such that every word $w \in A^*$ admits a factorization of the form $w = u_0v_1u_1...v_ku_k$ with $u_1, \ldots, u_{k-1} \in A^+$, $u_0, u_k \in A^*$, $v_1, \ldots, v_k \in A^+$ and*

(1) *$v_1\eta, \ldots, v_k\eta$ are regular elements of M,*

(2) *if* b_{i-1} *denotes the last letter of* u_{i-1} *and* a_i *the first letter of* u_i, $(b_i v_i)\eta$ *and* $(v_i a_i)\eta$ *are not regular,*

(3) $|u_0 \cdots u_k| \leqslant N$.

Recall the definition of the Green's relation $\mathcal{R}$. Let u and v be two elements of a monoid M. Then $u \,\mathcal{R}\, v$ if and only if there exist two elements $x, y \in M$ such that $ux = v$ and $vy = x$. An $\mathcal{R}$-class is regular if it contains a regular element. The following proposition summarizes the properties of $\mathcal{R}$-classes that are used in this paper.

Proposition 3.4 *Let* M *be a monoid with commuting idempotents. Then*

(1) *every regular* $\mathcal{R}$*-class* R *contains a unique idempotent* e,

(2) *for every* $x \in R$, $ex = x$,

(3) *for every* $u, v, s \in M$, $u \,\mathcal{R}\, v \,\mathcal{R}\, us$ *and* $us = vs$ *implies* $u = v$.

(Proof omitted)

Theorem 3.5 *A rational language* L *is accepted by a reversible automaton if and only if it satisfies conditions (a) and (b) or, equivalently, conditions (a) and (c).*

Proof. By Propositions 3.1 and **??**, it suffices to show that if L satisfies (a) and (c), then $L \in \mathcal{C}$. Let r be the maximum size of an $\mathcal{R}$-class of $M(L)$, and let N be the integer given by Proposition 3.3. Let $\mathcal{F}$ be the set of all the reversible automata of the form $\mathcal{B} = (Q, A, \cdot, I, F)$ where Q contains at most $r(N+1)$ states and the language accepted by $\mathcal{B}$ is contained in L. $\mathcal{F}$ is a finite set, since there are only a finite number of automata with at most $r(N+1)$ states. Let $\mathcal{A}$ be the disjoint union of all the automata of $\mathcal{F}$. Then $\mathcal{A}$ is a reversible automaton such that $L(\mathcal{A}) \subset L$. To prove that $L(\mathcal{A})$ is actually equal to L, it suffices to exhibit, for every word $w \in L$, a reversible automaton $\mathcal{B}$ of $\mathcal{F}$ such that $w \in L(\mathcal{B})$.

Put $m = w\eta$ and denote by $P(m)$ the smallest subset of $M(L)$ containing m and satisfying condition (c): for every $s, e, t \in M(L)$, with e idempotent, $set \in P(m)$ implies $st \in P(m)$. Now, since $m \in P$, $P(m)$ is contained in P and the language $L(m) = P(m)\eta^{-1}$ is contained in L.

We first assume that m is a regular element of $M(L)$. Then, by Proposition 3.4, the $\mathcal{R}$-class R of m contains a unique idempotent e and we can state

Lemma 3.6 *The language $L(m)$ is accepted by the reversible automaton $\mathcal{B} = (R, A, \cdot, \{e\}, \{m\})$, where, for every $a \in A$, $x \cdot a = x(a\eta)$ if $x(a\eta) \in R$ (undefined otherwise).*

Proof. Proposition 3.4 shows that $\mathcal{B}$ is reversible. Next, $L(\mathcal{B}) = S\eta^{-1}$ where $S = \{s \in M(L) \mid es = m\}$ and $L(m) = P(m)\eta^{-1}$. Therefore, it suffices to show that $S = P(m)$. First $m \in S$ by Proposition 3.4, and if $sft \in S$ for some $s, t \in M(L)$ and some idempotent f, then $es \ \mathcal{R} \ esf \ \mathcal{R} \ (es)f = (esf)f$. It follows that $es = esf$ by Proposition 3.4, whence $est = esft = m$ and $st \in S$. Thus S satisfies condition (c) and $P(m)$ is contained in S. Conversely, let $s \in S$. Then $es = m$, and hence $1es = m \in P(m)$. Therefore, by condition (c), $s = 1.s \in P(m)$ and thus $S = P(m)$ as required.

We now turn to the general case. Let $w = u_0 v_1 u_1 \cdots v_k u_k$ be a factorization of w given by Proposition 3.3. Put, for $1 \leqslant i \leqslant k$, $v_i\eta = m_i$, and let e_i be the (unique) idempotent of the $\mathcal{R}$-class of m_i. The previous proposition shows that the language $L(m_i)$ is accepted by the automaton $\mathcal{B}_i = (R_i, A, \cdot, e_i, m_i)$.

We consider also the minimal automaton $\mathcal{B}$ of the word $u = u_0 u_1 \cdots u_k$ defined as follows. The set of states is the set of left factors of u and, for each letter $a \in A$ and for each left factor x of u, $x \cdot a = xa$ if xa is a left factor of u and is undefined otherwise. We now "sew" the automata $\mathcal{B}$ and $\mathcal{B}_i$'s together, according to the following diagram:

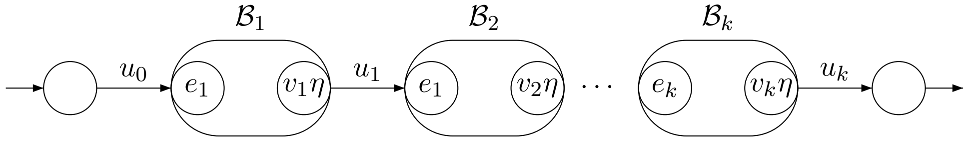

Figure 3.2: Sewing $\mathcal{B}$ and the $\mathcal{B}_i$'s together

Now, Proposition 3.3 implies that the resulting automaton is reversible (the details are omitted), accepts the language

$$K = u_0 L(m_1) u_1 \cdots u_{k-1} L(m_k) u_k$$

and contains at most $r(N+1)$ states. We claim that K is contained in $L(m)$ (and thus in L). Indeed put, for $0 \leqslant i \leqslant k$, $s_i = u_i\eta$, so that $m = s_0 m_1 s_1 \cdots m_k s_k$. Since $K \subset K\eta\eta^{-1}$, it suffices to show that $K\eta = s_0 P(m_1) s_1 \cdots P(m_k) s_k$ is contained in $P(m)$. Let T be the set of all $(t_1, \ldots, t_k)$ of $P(m_1) \times \cdots \times P(m_k)$ such that $s_0 t_1 s_1 \cdots s_k t_k \in P(m)$. Then T contains $(m_1, \cdots, m_k)$. Furthermore, if $(t_1, \cdots, t_k) \in T$ and if

$t_i = x_i f_i y_i$ for some idempotent f_i, then $(s_0 t_1 \cdots s_{i-1} x_i) f_i (y_i s_i ... s_k t_k) \in P(m)$, and hence, by Condition (c), $s_0 t_1 \cdots s_{i-1} x_i y_i s_i \cdots s_k t_k \in P(m)$, so that $(t_1, \ldots, t_{i-1}, x_i y_i, t_{i+1}, \ldots, t_k) \in T$. Therefore T is equal to $P(m_1) \times \cdots \times P(m_k)$ and this concludes the proof. □

4 A topological characterization.

In this section, we give a topological description of the class $\mathcal{C}$. Let us first recall the definition of the topology we are concerned with.

One can show that two distinct words u and v of A^* can always be separated by a finite group in the following sense: there exists a finite group G and a monoid morphism $\varphi : A^* \to G$ such that $\varphi(u) \neq \varphi(v)$. Set, for every $u, v \in F(A)$,

$$r(u, v) = \min \{\mathrm{Card}(G) \mid G \text{ is a finite group that separates } u \text{ and } v\}$$

and

$$d(u, v) = e^{-r(u,v)}$$

with the usual conventions $\min \emptyset = \infty$ and $e^{-\infty} = 0$. Then d is a distance (in fact an ultrametric distance) which defines a topology on A^*, called the *finite-group topology* of the free monoid. This topology, introduced by Reutenauer [14, 15], is an analogue for the free monoid to the profinite topology of the free group introduced by M. Hall [5]. It is the coarsest topology such that every monoid morphism from A^* into a discrete finite group is continuous. The free monoid A^*, equipped with this topology, is a topological monoid. The interested reader is referred to [12, 14] for a more detailed study of this topology. An example of a converging sequence is given by the following proposition:

Proposition 4.1 ([14]) *For every word $w \in A^*$, $\lim_{n\to\infty} w^{n!} = 1$.*

The next proposition relates reversible automata to this topology.

Proposition 4.2 ([14]) *Every language accepted by a reversible automaton is closed in the profinite group topology.*

The converse is not true in general. For instance, the language a^*b^* is closed but is not accepted by any reversible automaton. However, we have

Theorem 4.3 *A rational language L is accepted by a reversible automaton if and only if the idempotents commute in $M(L)$ and L is closed in the finite group topology.*

Proof. If L is accepted by a reversible automaton, then L is closed by Proposition 4.2 and the idempotents of $M(L)$ commute by Proposition 3.1. Conversely, if L is closed, then L satisfies (b). Indeed, let x, u, y be words such that $xu^+y \subset L$. Then, in particular, for every $n > 0$, $xu^{n!}y \in L$. Since L is closed, and since the multiplication is continuous, it follows by Proposition 4.1 that $xy = \lim_{n\to\infty} xu^{n!}y \in L$. Thus L satisfies (a) and (b) and the result follows from Theorem 3.5. □

We have conjectured [11] that a rational language is closed if and only if it satisfies condition (c) or (b). Theorem 4.3 shows that this conjecture is true if the idempotents of the syntactic monoid of L commute.

5 Summary.

Let us summarize the previous results into a single statement.

Theorem 5.1 *Let L be a rational language. The following conditions are equivalent:*

(1) *L is accepted by a reversible automaton,*

(2) *$L = K \cap A^*$ where K is a subset of the free group $F(A)$ consisting of a finite union of left cosets of finitely generated subgroups of $F(A)$,*

(3) *the idempotents of $M(L)$ commute and, for every $x, u, y \in A^*$, $xu^+y \in L$ implies $xy \in L$,*

(4) *the idempotents of $M(L)$ commute and, for every $s, t, e \in M$ such that e is idempotent, $set \in P$ implies $st \in P$,*

(5) *the idempotents of M commute and L is closed in the finite group topology of A^*.*

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